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Computing Homogeneous Least Squares Regression Equations using a Non-Homogeneous Program

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Our multiple regression program uses O.L.S. to find $\hat{\beta}$, a (kxl) vector of estimates of regression coefficients β, in the equation

$$y = \ell\alpha + X\beta + u \qquad u \sim N(O, \sigma^2 I_T) \qquad (1)$$

y is an (Txl) column vector of observations on the dependent variable,

ℓ is an (Txl) column vector in which each element has the value 1,

X is an (Txk) matrix of observations on k explanatory variables, and

n is an (Txl) column vector of random residuals.

However, we required to estimate a model where the intercept coefficient was known to have the value zero. Having insufficient time to rewrite our program, we thought that the desired result might be achieved by an unconventional alternative method, namely by systematically adjusting our data!

For each original datum X_{ij}, there was added the value $-X_{ij}$. By this adjustment the mean of each independent variable became zero; the regression line was thereby diverted to a path through the origin. We next considered the properties of our estimates.

Our required model was

$$y = X\beta + \varepsilon \qquad \varepsilon \sim N(O, \sigma^2 I_T) \qquad (2)$$

Therefore, the O.L.S. estimator of β, say $\hat{\beta}$, is

$$\hat{\beta} = (X'X)^{-1} X'y \qquad (3)$$

However, the computor program uses (1) to estimate β, say $\bar{\beta}$

$$\bar{\beta} = (\bar{X}'\bar{X})^{-1} \bar{X}'\bar{y} \qquad (4)$$

where $\bar{X}$, $\bar{y}$ denotes the observations expressed as derivations from means.

Consider our adjusted information matrix

$$\begin{pmatrix} y & \vdots & X \\ \cdots & & \cdots \\ -y & \vdots & -X \end{pmatrix} \qquad (5)$$

The estimate of β from (4) then becomes

$$\begin{pmatrix} \tilde{\alpha} \\ \tilde{\beta} \end{pmatrix} = \left\{ \begin{pmatrix} \ell' & \vdots & \ell' \\ \cdots & \cdots & \cdots \\ X' & \vdots & -X' \end{pmatrix} \begin{pmatrix} \ell & \vdots & X \\ \cdots & \cdots & \cdots \\ \ell & \vdots & -X \end{pmatrix} \right\}^{-1} \begin{pmatrix} \ell' & \vdots & \ell' \\ \cdots & \cdots & \cdots \\ X' & \vdots & -X' \end{pmatrix} \begin{pmatrix} y \\ \cdots \\ -y \end{pmatrix}$$

$$= \left\{ \begin{matrix} 2\ell'\ell & \vdots & O \\ \cdots & \cdots & \cdots \\ O & \vdots & 2X'X \end{matrix} \right\}^{-1} \begin{pmatrix} O \\ \cdots \\ 2X'y \end{pmatrix}$$

Since the first matrix on the R.H.S. is block diagonal, it follows that

$$\tilde{\beta} = \frac{1}{2}(X'X)^{-1}\, 2X'y \qquad (6)$$

$$= (X'X)^{-1}\, X'y$$

$$= \hat{\beta}$$

and

$$\tilde{\alpha} = O.$$

By using (1), the program will also give the following estimate

$$\text{Var } \tilde{\beta} = \frac{S^2}{2}\,(X'X)^{-1} \qquad (7)$$

where

$$S^2 = \frac{2\Sigma e^2}{2T - k - 1}\,. \qquad (8)$$

We know the "correct" estimate of σ^2 is

$$\hat{\sigma}^2 = \frac{\Sigma e^2}{T - k} = \frac{1}{2} \cdot \frac{S^2}{T - k} \cdot 2T - k - 1 \qquad (9)$$

So to obtain the correct estimate, Var $\hat{\beta}$, we adjust our program's own estimate, Var $\tilde{\beta}$ by the factor $2T - k - 1/T - k$, i.e.

$$\text{Var } \tilde{\beta} \cdot \frac{2T - k - 1}{T - k} = \frac{S^2}{2}\,(X'X)^{-1} \cdot \frac{2T - k - 1}{T - k}$$

$$= \frac{2\Sigma e^2}{2(2T - k - 1)}\,(X'X)^{-1}\,\frac{2T - k - 1}{T - k}$$

$$= \frac{\Sigma e^2}{T - k}\,(X'X)^{-1}$$

$$= \text{Var } \hat{\beta}.$$

The adjustments to the computer print-out are sufficiently simple to have so far precluded our writing a homogeneous program.

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